Recitation 9/24 Solutions

1. Consider the set $\left\{ \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}1\\-2\\-7\\0\end{bmatrix}, \begin{bmatrix}-1\\4\\12\\2\end{bmatrix} \right\}$.

   (a) Determine whether the set is linearly independent.

   Let $A = \begin{bmatrix}1 & 1 & -1\\2 & -2 & 4\\3 & -7 & 12\\4 & 0 & 2\end{bmatrix}$. The RREF of A is $\begin{bmatrix}1 & 0 & \frac{1}{2}\\0 & 1 & -\frac{3}{2}\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$. Since there is not a pivot in every column, the set is not linearly independent.

   (b) Find a basis for Span$\left\{ \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}1\\-2\\-7\\0\end{bmatrix}, \begin{bmatrix}3\\0\\1\\2\end{bmatrix} \right\}$. What is its dimension?

   We saw in class that the pivot columns of A form a basis for A. Since the first two columns are the pivot columns, we have that $\left\{ \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}1\\-2\\-7\\0\end{bmatrix} \right\}$ forms a basis for A.

2. Let $A = \begin{bmatrix}1 & 2 & 0 & 4\\-1 & -3 & 7 & 8\\5 & 2 & 1 & 6\end{bmatrix}$. Find a basis for the column space and null space of A.

   First we reduce A to RREF: $\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 2\\0 & 0 & 1 & 2\end{bmatrix}$.

   Since the first three columns are pivto columns, $\left\{ \begin{bmatrix}1\\-1\\5\end{bmatrix}, \begin{bmatrix}2\\-3\\2\end{bmatrix}, \begin{bmatrix}0\\7\\1\end{bmatrix} \right\}$ is a basis for $C(A)$.

   In order to find a basis for $N(A)$, we must find the special solutions. The underlying system is

$$\begin{aligned} x_1 &= 0\\ x_2 + 2x_4 &= 0\\ x_3 + 2x_4 &= 0 \end{aligned}$$

Solving in terms of the free variable x_4, we have the general solution:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2x_4 \\ -2x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ -2 \\ -2 \\ 1 \end{bmatrix}$$

Hence $\left\{ \begin{bmatrix} 0 \\ -2 \\ -2 \\ 1 \end{bmatrix} \right\}$ forms a basis of $N(A)$.

3. In class, we saw (in some examples) that a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of vectors in $\mathbb{R}^n$ is linearly dependent if and only if one of the vectors is a linear combination of the others. Let's prove this fact in two steps.

   (a) Show that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent, then one of the vectors is a linear combination of the others.

   Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be linearly dependent. Then there are weights $c_1, \ldots, c_p$ not all equal to 0 such that

   $$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

   Hence one of the weights is nonzero; say $c_i \neq 0$. Then we can rewrite the above equation. First, isolate the term $c_i\mathbf{v}_i$:

   $$c_i\mathbf{v}_i = -c_1\mathbf{v}_1 - \cdots - c_{i-1}\mathbf{v}_{i-1} - c_{i+1}\mathbf{v}_{i+1} - \cdots - c_p\mathbf{v}_p$$

   Multiplying both sides by $\frac{1}{c_i}$ (which we can do since $c_i \neq 0$) gives:

   $$\mathbf{v}_i = -\frac{c_1}{c_i}\mathbf{v}_1 - \cdots - \frac{c_{i-1}}{v_i}\mathbf{v}_{i-1} - \frac{c_{i+1}}{c_i}\mathbf{v}_{i+1} - \cdots - \frac{c_p}{c_i}\mathbf{v}_p.$$

   In other words, $\mathbf{v}_i$ is a linear combination of the other vectors.

   (b) Show that if one of the vectors in $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a linear combination of the others, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

   Suppose $\mathbf{v}_i$ is a linear combination of the other vectors. Then

   $$\mathbf{v}_i = d_1\mathbf{v}_1 + \cdots + d_{i-1}\mathbf{v}_{i-1} + d_{i+1}\mathbf{v}_{i+1} + \cdots + d_p\mathbf{v}_p$$

   for some weights $d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_p \in \mathbb{R}$. By subtracting $\mathbf{v}_i$ from both sides, we obtain the equation

$$d_1\mathbf{v}_1 + \cdots + d_{i-1}\mathbf{v}_{i-1} + (-1)\mathbf{v}_i + d_{i+1}\mathbf{v}_{i+1} + \cdots + d_p\mathbf{v}_p = \mathbf{0}$$

In other words, there exists a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with weights not all zero yielding $\mathbf{0}$. Thus $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

4. **Looking Ahead** Let A be an $n \times n$ matrix. Explain why the following statements are equivalent.
   (a) A is invertible
   (b) $N(A) = \{\mathbf{0}\}$
   (c) The columns of A are linearly independent

   A is invertible if and only if A has n pivots. Since A is $n \times n$, this is equivalent to saying A has a pivot in each column. Now A has a pivot in each column if and only if $A\mathbf{x} = \mathbf{0}$ has only the zero solution (i.e. there are no free variables); this means that $N(A) = \{\mathbf{0}\}$. We also know that A has a pivot in each column if and only if the columns of A are linearly independent.